Nonlinear Resonance as the Cause of Multiple Pure Tones

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When the fans of aeroengine ducts go supersonic, they often produce radiation at the subharmonics of blade-passage frequency, known as multiple pure tones (MPT). It has been shown that the conventional explanation, that these MPT's are created by the shock waves, is inadequate. An alternative mechanism based on the concept of a "strong interaction" between the harmonics is proposed. Expression for the governing equation for such an interaction is derived. The results show an improved agreement with observed data. The analysis has also led to several practical suggestions for a suppression of the noise.

Nomenclature

a	= inner radius of the annular duct
$\bar{a}$	= mean radius of the annular duct
a_j	= phase shift of jth harmonic
$\bar{a}_j$	= mode correction in phase in jth harmonic
A_j	= amplitude of jth harmonic
$A_m(\ )$	= amplitude modulation, see Eq. (24)
$\bar{A}_j$	= mode correction in amplitude of jth harmonic
b	= outer radius of the annular duct
$\bar{b}$	= constant, defined as $\log_\eta(1+\epsilon_2)/\epsilon_2$
B	= number of blades
c_0	= speed of sound at the stagnation point
E_j	= depletion function, see Eq. (47)
f_j	= dissipation coefficient, see Eq. (52)
G_j	= generation function, see Eq. (46)
h	= width of the annular duct, $= b - a$
$L(\ )$	= wave operator, $\equiv c_0 \nabla^2 - \partial^2/\partial t^2$
$L_{m_1}(\ )$	= mode correction I, see Eq. (30)
$L_{m_2}(\ )$	= mode correction function II, see Eq. (31)
m	= mode number in radial direction
$M_m(\ )$	= mode modification, see Eq. (28)
n	= spinning mode
$p,\bar{p}$	= acoustic pressure
q	= spinning correction, see Eqs. (47a, b)
$r,\bar{r}$	= radial coordinate
R_0	= resistance of the lining of duct
s_j	= shape factor, see Eq. (44)
$t,\bar{t}$	= time
w	$= \psi/\epsilon_2$
w_0	$= w/\bar{b}$
$x,\bar{x}$	= radial coordinate
$z,\bar{z}$	= axial direction of the duct
z_1	= multiple scale in z direction, $\equiv \epsilon z$
$\alpha,\bar{\alpha}$	= wave number of z direction
β	= a parameter, see Eq. (41)
γ	= specific-heat ratio
ϵ	= particle Mach number
ϵ_2	$\equiv h/a$
η_2	$\equiv \epsilon_2/\epsilon$
θ	= propagating phase
$\phi,\bar{\phi}$	= velocity potential
χ_0	= reactance of the lining of duct
ψ	= azimuthal angle
$\bar{\psi}_j$	= second-order solution in jth harmonic
$\omega,\bar{\omega}$	= frequency
ω_n	= blade-passage frequency
Ω	= shaft frequency
$\bar{p},\bar{r},\bar{t},\bar{z},\bar{\phi}$	= dimensional quantities, these same symbols without overbars are nondimensional

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I. Introduction

AEROENGINE fans radiate the multiple pure tones (MPT's), essentially only at the supersonic tip speeds. At subsonic tip speeds their radiation is much easier to understand. Most of the discrete frequency radiation is at the blade-passage frequency and its harmonics. The blade-passage frequency is equal to the shaft rotational frequency multiplied by the number of blades. Thus a 16-blade rotor with a shaft rotating at 1000 rev/sec generates radiation at 16,000 Hz, 32,000 Hz, 48,000 Hz, etc. Only if the blades are unevenly spaced, would radiation at the direct multiples of shaft rotational frequency (i.e. at 1000 Hz, 2000 Hz, etc., in this example) be observed.

Once the tip speed goes supersonic, however, the radiation at the multiples of shaft frequency or at the subharmonics of the blade-passage frequency becomes a rule rather than an exception. Based on the linear theory, the amplitude of these subharmonics is much larger than could be accounted for by the nonuniformity of blade spacing. In any case, the linear theory could not possibly explain the observation that a probing of the near field of the rotor often shows that these MPT's grow in their amplitude as they propagate upstream in the engine inlet. This observation, along with other interesting characteristics of the MPT's stated in Sec. IIB, needs to be explained adequately.

When the tip speeds go supersonic, shock waves are generated due to the motion of the blades. It is now widely believed that these shock waves are responsible for the MPT generation. An explanation based on this hypothesis, given by Hawkings,[2] gives a qualitative explanation of the phenomenon. However, it fails to explain several of the observed data. The inadequacy of the weak shock theories is outlined in Sec. IIC.

It has been known that the MPT's propagate in the engine duct in the form of spinning modes. On one hand, the weak shock theories try to explain MPT's essentially in terms of plane waves. On the other hand, conventional linear duct acoustics cannot explain the phenomenon. There is a need to develop a three-dimensional nonlinear propagation theory. Such a theory could indirectly account for the possible presence of shocks through spatial Fourier transforms, provided proper techniques for nonlinear superposition could be developed.

We have found that the linear theory is inadequate near some critical wave numbers. This would be true even if the intensity of sound were to be relatively low. Most of the nonlinear theories in the past have been restricted to the propagation of plane waves. Therefore this problem of critical wave numbers has been left largely unexplored. It has been shown by the present authors[3,4] that in the case of a rigid or slightly soft rectangular duct, a "strong interaction" exists between various modes of the duct. This interaction is stronger the nearer the wave number is to the cutoff wave number. We have also explained the generation of subharmonics to be due to such an interaction.

The case of an annular duct is more complex. We propose to show that similar strong interactions can exist in such ducts. To account for such possible interactions, we begin Sec. III with the nonlinear wave equation. This equation is stated in the cylindrical coordinates. We then seek an asymptotic form of this equation. For a very narrow annulus, the form of this equation is identical with that for a rectangular duct. From our previous work, it is clear that this case would display a strong resonance (physically the resonance is manifested when the eigenvalues are integral multiplies of one another). If the annulus is not very narrow, the deviation from the rectangular form is manifest as a perturbation of the governing equation.

The resonance expansion method[3] was developed to analyze problems similar to the one in this paper. Using this method, a uniformly valid first-order solution could be obtained by eliminating the secular terms in the second-order analysis. The analysis shows the presence of strong interactions near the critical tip Mach number. These interactions are seen to be weakened by a slight amount of softness introduced on the wall near the rotor.

II. Review of Previous Works

A. Propagation of the Radiation of Subsonic Rotors

For a narrow annulus, Tyler and Sofrin's theory[1] can be stated as follows: for a rotor with B blades, rotating at a speed of Ω rad/sec, a pressure field $p(\psi)$ should exist in front of it. If the incoming flow is uniform and the blades are evenly spaced, the field at any instance could be represented by

$$p=\sum_n B_n \cos(nB\psi + a_n) \qquad \text{at } \bar{t}=\bar{z}=0 \tag{1}$$

Since this field rotates as a whole at the shaft rotational speed Ω rad/sec, the dependance on time could be included as

$$p=\sum_n B_n \cos[nB(\psi-\Omega t)+a_n] \qquad \text{at } \bar{z}=0 \tag{2}$$

With this initial condition, the field upstream of the rotor is required. In the absence of mean flow, if the nonlinear terms are neglected, the wave equation is

$$\frac{1}{c_0}\frac{\partial^2 \bar{p}}{\partial \bar{t}^2}=\frac{\partial^2 \bar{p}}{\partial \bar{z}^2}+\frac{\partial^2 p}{\bar{a}^2 \partial \psi^2} \tag{3}$$

For outgoing waves, the solution could be assumed to be of the form

$$\bar{p}=\sum_n B_n \cos(\bar{\alpha}_n \bar{z}-\bar{\omega}_n \bar{t}+nB\psi+a_n) \tag{4}$$

The wave equation and the form of the initial conditions give rise to the following relationships:

$$\bar{\omega}_n = nB\bar{\Omega} \tag{5}$$

$$\bar{\alpha}_n{}^2=\frac{\omega_n{}^2}{c_0{}^2}-\frac{n^2B^2}{\bar{a}^2} \tag{6}$$

The solution can be interpreted as a set of spiralling waves since the dependence on time is through $(\psi-\Omega t)$. It is also seen that when $\omega_n/c_0<nB/\bar{a}$, $\bar{\alpha}_n$ becomes imaginary. In this case, the wave decays in amplitude as it propagates. In terms of $\bar{\Omega}$, this condition occurs when

$$\frac{\bar{\Omega}\bar{a}}{c_0}<1 \tag{7}$$

i.e., when tip Mach number is less than unity. It is interesting and relevant to the problem in the next section that the cutoff frequency of all the modes is reaches simultaneously when the rotor tip speed is supersonic, although this condition gets somewhat modified due to the presence of stators or nonuniformities.

Equation (5) shows that the various spinning modes are generated at frequencies $B\Omega$ times the modal numbers. It will be seen that this is indeed the precondition for strong resonance. The modification of the Tyler-Sofrin theory in Sec. III will be based on these observations.

B. Characteristics of Multiple Pure Tones

Data from actual fans[7-11] reveal the following characteristics:

1) MPT's are detected in most cases in the inlet duct and the forward-arc far-field noise data.

2) They are observed only when the rotor tip Mach number is greater than unity.

3) The highest sound-pressure levels exist, in general, below the blade-passage frequency at integral multiples of the shaft rotational frequency (Fig. 1).

4) The level of the MPT's is low right next to the rotor and increases upstream of it.

5) MPT levels tend to increase rapidly up to a Mach number of 1.2-1.3, dominated by frequencies near half the blade-passing frequency, then decrease in intensity as the Mach number increases.

6) The intensity of blade-passing-frequency tone decays to the minimum at some distance from the blade; afterward it increases.

7) Only a small amount of absorbent treatment, and of a value different from that indicated by conventional optimization techniques, applied on the duct wall near the rotor suppresses MPT's.

An adequate theoretical framework is needed to explain all of the above characteristics and to predict further useful results. The conventional theory, summarized in the next subsection, explains only a few of these.

C. Weak-Shock Theory

It is well known that once the relative tip Mach number of the blades goes supersonic, the leading edges would shed a series of shock waves. These shocks are spaced at the same distance as the blades. If the blade spacing is uniform, they propagate in a spiral manner. In the frequency domain, the signal they carry would contain only the blade-passage frequency and its harmonics.

If the strength of the formed shock is small, we can ignore the effects of entropy generation, which varies as the third power of the strength. This simplification yields a powerful method, called the weak-shock theory, to deal with the propagation of these shocks. Hawkings[2] used a technique that combined geometric acoustics and weak-shock theory to study the transmission of shocks. It was shown that the upstream acoustic field depends primarily upon a single-stage parameter that was defined by a simple integral of the duct area and the velocity distribution.

Hawkings also showed that if the blade spacing is slightly nonuniform or if the initial shock strengths are slightly different, the nonuniformity in the strength of the shocks increases. This nonuniformity in frequency domain corresponds to MPT's at the submultiples of the blade-passage frequency.

D. Limitations of the Weak-Shock Theory

The weak-shock theory adequately explains the characteristics designated 1-4 in Sec. IIB. This agreement has led to a consensus so strong that the inadequacy of the explanation for characteristics 5-7 is almost totally ignored by most researchers.

It is clear that the effect of absorption on the MPT's would involve a three-dimensional analysis; so would the study of the effect of shear and geometry of cross section. These effects are not dealt with adequately by means of an essentially one-dimensional method.

In fairness it should be added that the mechanism we propose has some epistemologic overlap with the weak-shock explanation. The shock dispersion occurs due to the nonlinear acoustic-acoustic interaction. The same interaction is responsible for the resonance phenomenon[5] that we offer as an alternative mechanism. However, to regard the shocks as the cause of the MPT's is erroneous.

III. Propagation of the Radiation of Supersonic Fans in an Annulus

A. Introduction

MPT's can exist with and without shocks. When the shocks are present, ideally we would like a three-dimensional theory of shock propagation in a lined duct carrying sheared flow. Since this problem is enormously complicated, we must extract the aspect of the problem that is crucial for prediction of observed phenomena. As a first step, we represent the field locally in Fourier domain. If weak shocks are to be present, we can take advantage of the cubic generation of entropy and deal only with vorticity and acoustic modes. This argument forms the basis of our theory.

B. Reduction of the Wave Equation

Initially, we will limit our considerations to the nonlinear interactions of the acoustic-acoustic type. The governing equation for the velocity potential $\bar{\phi}$ is

$$c_0 \nabla^2 \bar{\phi} - \frac{\partial^2 \bar{\phi}}{\partial \bar{t}^2} = (\gamma - 1) \frac{\partial \bar{\phi}}{\partial \bar{t}} \nabla^2 \bar{\phi} + 2 \nabla \bar{\phi} \cdot \nabla \frac{\partial \bar{\phi}}{\partial \bar{t}} + \frac{(\gamma-1)}{2} (\nabla \bar{\phi})^2 \nabla^2 \bar{\phi} + \frac{1}{2} (\nabla \bar{\phi} \cdot \nabla)(\nabla \bar{\phi})^2 \tag{8}$$

The procedure of nondimensionalization for this problem is carried out below. The derivative operators are defined by

$$\frac{\partial}{\partial \bar{r}} = \frac{1}{h}\frac{\partial}{\partial r} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{h}\frac{\partial}{\partial z} \quad \frac{\partial}{\partial \bar{\psi}} = \frac{a}{h}\frac{\partial}{\partial w} \quad \frac{\partial}{\partial \bar{t}} = \frac{c_0}{h}\frac{\partial}{\partial t} \tag{9}$$

The velocity potential $\bar{\phi}$ and radial distance $\bar{r}$ are

$$\bar{\phi} = \epsilon c_0 h \phi \quad \text{and} \quad \bar{r} = ar \tag{10}$$

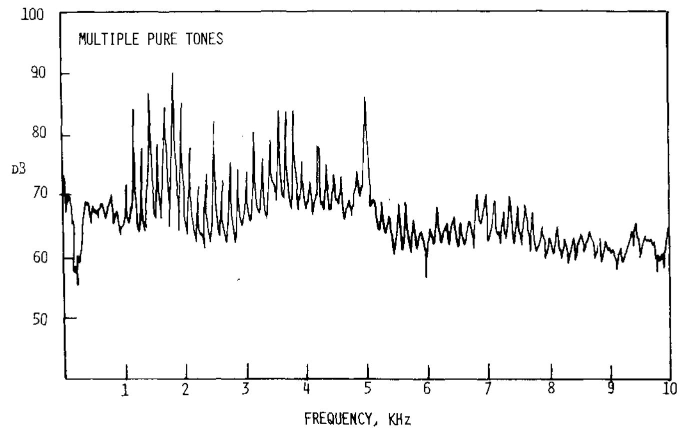

Fig. 1 Typical spectrum containing multiple pure tones – far-field data.

where ϵ is the particle Mach number. Substituting Eqs. (9) and (10) into Eq. (8) yields the governing equation for ϕ:

$$L(\phi) = \epsilon N_1(\phi) + \epsilon^2 N_2(\phi) \tag{11}$$

where L is the wave operator, defined as $\nabla^2 - \partial^2/\partial t^2$. The nonlinear operators N_1 and N_2 are

$$N_1(\phi) = (\gamma - 1)\phi_t \nabla^2 \phi + 2 \nabla \phi \cdot \nabla \phi_t \tag{12}$$

and

$$N_2(\phi) = \frac{(\gamma-1)}{2} (\nabla \phi)^2 \nabla^2 \phi + \frac{1}{2} (\nabla \phi \cdot \nabla)(\nabla \phi)^2 \tag{13}$$

The subscript t represents the time derivative. The nondimensional Laplacian and gradient operators are

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \epsilon_2 (1/r) \frac{\partial}{\partial r} + (1/r^2) \frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial z^2}$$

and

$$\nabla = \frac{\partial}{\partial r} i + (1/r) \frac{\partial}{\partial w} j + \frac{\partial}{\partial z} k$$

The parameter ϵ_2 in the Laplacian operator is defined as the ratio h/a. The magnitude of ϵ_2 indicates the relative importance of the term $(1/r)\partial/\partial r$. If ϵ_2 is small (e.g., narrow annular ducts), we can put the term $(1/r)\partial\phi/\partial r$ from the left-hand side of Eq. (11) to the right-hand side as a correcting term that perturbs the modes of the input waves. Furthermore, by the analysis in Ref. 5, the variable coefficients in Laplacian and gradient operators, i.e. $1/r$ and $1/r^2$, could be replaced by $\bar{b}$, $\log_\eta = (1+\epsilon_2)/\epsilon_2$, and $\bar{b}^2$, respectively. We can classify the annular ducts into two types according to the magnitude of ϵ_2. The classifications and their corresponding governing systems are:

Large annular ducts, $O(\epsilon_2) = 1$:

$$L(\phi) = \left(\frac{\partial^2}{\partial r^2} + \epsilon_2 \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} \right)\phi = \epsilon N_1(\phi) + \epsilon^2 N_2(\phi) \tag{14}$$

Narrow annular ducts, $O(\epsilon_2) < 1$:

$$L(\phi) = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} \right)\phi = -\epsilon_2 (L_{m_1}(\phi) + L_{m_2}(\phi)) + \epsilon N_1(\phi) + \epsilon^2 N_2(\phi) \tag{15}$$

where

$$w_0 = w/\bar{b} \tag{16}$$

$$L_{m_1}(\phi) = \frac{1}{r}\frac{\partial\phi}{\partial r} \tag{17}$$

$$L_{m_2}(\phi) = \epsilon_2^{-1}\left(\frac{1}{\bar{b}^2 r^2}\frac{\partial^2\phi}{\partial w_0^2} - \frac{\partial^2\phi}{\partial w_0^2}\right) \tag{18}$$

The boundary conditions (for a rigid-walled duct) are

$$r = \frac{1}{\epsilon_2} \quad \phi_r = 0 \quad \text{at the inner boundary} \tag{19}$$

$$r = 1 + \frac{1}{\epsilon_2} \quad \phi_r = 0 \quad \text{at the outer boundary} \tag{20}$$

Equation (15) is of the same form as the original equation for ϕ. The only change is that we have put the small terms with variable coefficients on the right-hand side. These terms modify the cross modes of a rectangular duct to the one satisfying a narrow annular duct.

C. Solution for the Resonant Case

If ϵ is small, we can use the perturbation technique to deal with the problem. Let

$$\phi = \phi_1 + \epsilon\phi_2 + \epsilon^2\phi_3 + \ldots \tag{21}$$

the first approximation of the solution ϕ in an annular duct is obtained by letting the left-hand side of Eq. (14) or (15) be zero. Thus, if ϕ_l is the solution of the linear equation, ϕ_1 might be assumed to be the same as ϕ_l. In that case, the equation for ϕ_2 would be written in terms of this ϕ_l. If this equation had no solutions of the order of $1/\epsilon$, the scheme of Eq. (21) would be successful. It turns out that this straightforward method does not work. Substitution of ϕ_l for ϕ_1 leads to secular terms in the second-order equation for the evaluation of ϕ_2. The presence of the secular terms indicates that the system is in resonance.

It has been known that the first-order linear solutions would interact to give rise to second-order terms. Such an interaction might be termed the "weak interaction." In the case under discussion, those terms interact at the first-order itself. We have chosen to term this the "strong interaction." Following the technique of resonance expansion, ϕ_1 is expressed as

$$\phi_1 = \sum_{j=1}^{\infty} A_j(z_1)\,[1 + \epsilon_2\bar{A}_j(x)]\cos[jm\pi x + \epsilon_2^3\bar{a}_m(\bar{x})] \cdot \cos[j\theta + a_j(z_1)] \tag{22}$$

where $z_1 = \epsilon z$ and $\bar{x} = x$. The amplitudes A_j and phase shifts a_j are determined by eliminating the secular terms containing $N_1(\phi_1)$. The mode corrections A_j and $\bar{a}_j$ could be found by second-order analysis.

Now the second-order system becomes

$$L(\phi_2) = [A_m + N_1(\phi_1)] + \eta_2[M_m - L_m(\phi_1)] \tag{23}$$

where $\eta_2 = \epsilon_2/\epsilon$. The modulation in amplitude A_m is

$$A_m \equiv -2\frac{\partial^2\phi_1}{\partial z\partial z_1} = 2\sum_{j=1}^{\infty} j\dot{A}_j(1 + \epsilon_2\bar{A}_j)\cos(jm\pi x + \epsilon_2^3\bar{a}_j) \times \sin(j\theta + a_j) + 2\alpha\sum_{j=1}^{\infty} jA_j\dot{a}_j(1 + \epsilon_2\bar{A}_j) \times \cos(jm\pi x + \epsilon_2^3\bar{a}_j)\cos(j\theta + a_j) \tag{24}$$

where the overdot denotes the derivative with respect to z_1. The forcing term N_1, representing the interaction of waves, is

$$N_1(\phi_1) = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\bar{F}_1(x)jl^2A_jA_l(1 + \epsilon_2\bar{A}_l)\sin[(j+l)\theta + a_j + a_l] + \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\bar{F}_2(x)jl^2A_jA_l(1 + \epsilon_2\bar{A}_j)(1 + \epsilon_2\bar{A}_l)\sin[(j-l) + a_j - a_l] \tag{25}$$

The functions $\bar{F}_1(x)$ and $\bar{F}_2(x)$ are

$$\bar{F}_1(x) = (-\tfrac{1}{2}(\gamma-1)\omega^3 - \alpha^2\omega - \epsilon_2^2 n^2\omega)\cos(jm\pi x + \epsilon_2^3\bar{a}_j) \times \cos(lm\pi x + \epsilon_2^3\bar{a}_l) + m^2\pi^2\omega\sin(jm\pi x + \epsilon_2^3\bar{a}_j) \times \sin(lm\pi x + \epsilon_2^3\bar{a}_l) \tag{26}$$

and

$$\bar{F}_2(x) = (-\tfrac{1}{2}(\gamma-1)\omega^3 - \alpha^2\omega - \epsilon_2^2 n^2\omega)\cos(jm\pi x + \epsilon_2^3\bar{a}_j) \times \cos(lm\pi x + \epsilon_2^3\bar{a}_l) - m^2\pi^2\omega\sin(jm\pi x + \epsilon_2^3\bar{a}_j) \times \sin(lm\pi x + \epsilon_2^3\bar{a}_l) \tag{27}$$

The modifications in the transverse modes M_m is

$$M_m \equiv -2\frac{\partial^2\phi_1}{\partial x\partial\bar{x}} = 2m\pi\sum_{j=1}^{\infty} jA_j[\bar{A}_j'\sin(jm\pi x + \epsilon_2^3\bar{a}_j) + \epsilon_2^2(1 + \epsilon_2\bar{A}_j)\bar{a}_j'\cos(jm\pi x + \epsilon_2^3\bar{a}_j)]\cos(j\theta a_j) \tag{28}$$

where the prime denotes the derivative with respect to $\bar{x}$. The mode correction term $L_m(\phi_1)$ is defined as

$$L_m(\phi_1) = L_{m_1}(\phi_1) + L_{m_2}(\phi_1) \tag{29}$$

where

$$L_{m_1}(\phi_1) = \frac{1}{1+\epsilon_2\bar{x}}\frac{\partial\phi_1}{\partial x} = \frac{-m\pi}{1+\epsilon_2\bar{x}}\sum_{j=1}^{\infty} \times jA_j(1 + \epsilon_2\bar{A}_j)\sin(jm\pi x + \epsilon_2^3\bar{a}_j)\cos(j\theta + a_j) \tag{30}$$

and

$$L_{m_2}(\phi_1) = \frac{1}{\epsilon_2}\left(\frac{1}{(1+\epsilon_2\bar{x})^2\bar{b}^2} - 1\right)\frac{\partial^2\phi_1}{\partial w_0^2} = -\epsilon_2 n^2\left[\frac{1}{(1+\epsilon_2\bar{x})^2\bar{b}^2} - 1\right]\sum_{j=1}^{\infty} jA_j(1 + \epsilon_2\bar{A}_j)\cos(jm\pi x + \epsilon_2^3\bar{a}_j)\cos(j\theta + a_j) \tag{31}$$

If the input condition at $z=0$ contains only one harmonic wave, say, at $z=0$

$$A_j = \begin{matrix} 1 & (j \geq 2) \\ 0 & (j = 1) \end{matrix} \tag{32}$$

we can drop the variations in phase shifts a_j, i.e. $a_j = 0$ for all j.

In this case, let $L_m(\phi_1)$ be balanced by M_m. We have

$$M_m - L_m(\phi_1) = 0 \tag{33}$$

This yields the solutions of $\bar{A}_j$ and $\bar{a}_j$

$$\bar{A}_j = \frac{1}{\epsilon_2}\left((1+\epsilon_2\bar{x})^{-1/2} - 1\right) \qquad (j=1,2,\ldots) \tag{34}$$

and

$$\bar{a}_j = \frac{jn^2\bar{x}}{2m\pi\epsilon_2}\left(1 - \frac{1}{\bar{b}^2(1+\epsilon_2\bar{x})^2}\right) \qquad (j=1,2,\ldots) \tag{35}$$

Therefore the first-order solution is

$$\phi = \sum_{j=1}^{\infty} A_j(z_1)(1+\epsilon_2\bar{x})^{-1/2}\cos(jm\pi x + \epsilon_2^3\bar{a}_j)\cos j\theta \tag{36}$$

Now the solution ϕ_2 can be assumed to be of the form

$$\phi_2 = \sum_{j=1}^{\infty} \tilde{\psi}_j(x, z_1)\sin j\theta \tag{37}$$

Substituting Eqs. (25), (33), and (37) into (23) yields

$$\frac{\partial^2\tilde{\psi}_j}{\partial x^2} + j^2m^2\pi^2\tilde{\psi}_j = 2j\left[\frac{\alpha\dot{A}_j}{(1+\epsilon_2 x)^{1/2}} - \frac{\beta(G_j - E_j)}{1+\epsilon_2 x}\right]$$

$$\cdot\cos(jm\pi x + \epsilon_2^3\bar{a}_j) + \text{NST} \tag{38}$$

where NST stands for the nonsecular terms. The boundary conditions are

$$\frac{\partial\tilde{\psi}_j}{\partial x} = 0 \qquad \text{at } x=0 \text{ and } 1 \tag{39}$$

The necessary and sufficient condition that Eq. (38) with boundary conditions (39) has a solution is[12]

$$\dot{A}_j - \beta s_j(G_j - E_j) = 0 \tag{40}$$

where the parameter β is defined by

$$\beta = \frac{(\gamma+1)\omega}{8\alpha} \tag{41}$$

The generation functions G_j and the depletion functions E_j are

$$G_j = \sum_{l=1}^{j} l(j-l)A_{j-l}A_l + \frac{\eta}{2}\left(\frac{j}{2}\right)^2 A_{j/2}^2 \tag{42}$$

and

$$E_j = \sum_{l=1}^{\infty} l(j+l)A_{j+l}A_l \tag{43}$$

S_j might be called the shape factors of the annular duct. They are given by

$$s_j = \frac{\int_0^1 (1+\epsilon_2 x)^{-1}\cos(jm\pi x + \epsilon_2^3\bar{a}_j)\cos jm\pi x\,dx}{\int_0^1 (1+\epsilon_2 x)^{-1/2}\cos(jm\pi x + \epsilon_2^3\bar{a}_j)\cos jm\pi x\,dx}$$

$$(j=1,2,\ldots) \tag{44}$$

These parameters indicate how the geometric shape of the duct influences the nonlinear propagation. As ϵ_2 increases beyond 1, the shape factors go to zero due to large $\epsilon_2^3\bar{a}_j$.

IV. A Mechanism for the Generation of the MPT's

A. Governing Equation for the Generation of Subharmonics

The interaction (40) could be rewritten as

$$\frac{dA_j}{dz} + \lambda_j(E_j - G_j) = 0 \tag{45}$$

where A_j is the nondimensional amplitude of various modes, and

$$\lambda_j = \epsilon\beta q s_j \qquad (j=1,2,\ldots) \tag{46}$$

where ϵ is the acoustic Mach number whose presence indicates that the higher the intensity of sound the higher are the nonlinear interactions. From the definition of β, it is seen that the nonlinearities are higher at higher frequencies and also that they are higher when $\alpha \to 0$. In fact, near the cutoff, even if ϵ is small the product $\epsilon\beta$ can be high. The cutoff frequency and the cutoff wave number α are influenced by the mean flow. Therefore β is also influenced by the mean flow. Furthermore,

$$q = 2 \qquad \text{(for spinning modes)} \tag{47a}$$

$$q = 1 \qquad \text{(for nonspinning modes)} \tag{47b}$$

E_j and G_j are the bilinear interaction terms that depend on the amplitudes of various modes.

B. Generation of Higher Harmonics

In Eq. (45) consider the case when initially only A_1 was nonzero. In this case, Eq. (45) shows that this harmonic will start spilling energy into higher harmonics. Each harmonic gets energy from the lower harmonics as indicated by the generation term G_j. Each harmonic loses energy to the higher harmonics, as indicated by the depletion term E_j. This problem was analyzed in the case of the rectangular duct,[6] which differs from our present case only in terms of possessing a different value of λ_j through q and s_j. It was shown there that if the initial value of a_1 was assumed to be zero, all the a_j remain zero all the time, for the single-input problem.

C. Generation of Subharmonics

Consider the case when, at $z=0$, two or more A_j are nonzero. If the A_j always remain in phase, we once again obtain Eq. (45), and only the higher harmonics are generated. If the initial inputs are not in phase, a more complicated situation arises. In the case of the rectangular duct, the general case is dealt with in Ref. 6. However, for the present problem, consider the case when all the initial phases are either 0 or 180 deg. We could still use Eq. (45) if we are willing to allow the A_j to be negative. Consider the particular case of the third harmonic, for instance. Equation (45) states that

$$\frac{1}{\lambda_3}\frac{dA_3}{dz} = 2A_1A_2 - 4A_1A_4 - 10A_2A_5 - 18A_3A_6 + \ldots \tag{48}$$

Consider further that A_8 is the harmonic corresponding to the blade-passage frequency. Let us assume an arbitrary value for A_8 equal to 1. Let A_5 be -0.001. This nonzero value could initially be assumed to be due to a slight nonuniformity of the blades. Let A_1, A_2, A_3, A_4, A_6, A_7, A_9, etc., be zero.

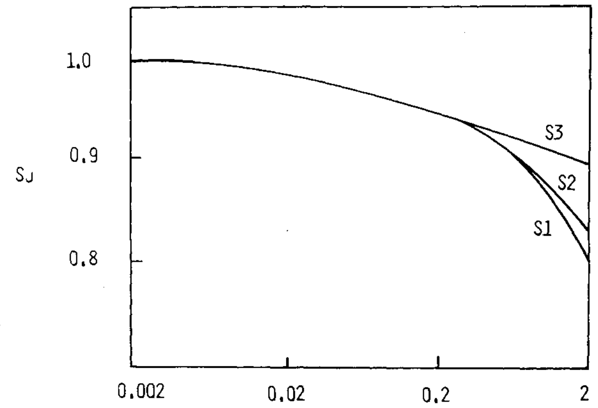

Fig. 2. Shape factors for $M=1$ and $N=4$.

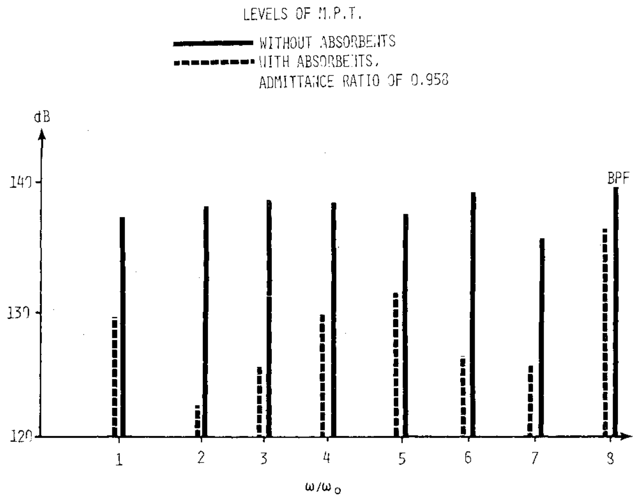

Fig. 3 Numerical simulation.

We would obtain for the initial rate of growth of A_3

$$\frac{1}{\lambda_3}\frac{\mathrm{d}A_3}{\mathrm{d}z}=+0.04 \tag{49}$$

that is A_3 would grow. This growth in turn will lead to the growth of other subharmonics.

It could be seen that as long as there is one dominant harmonic (at the blade-passage frequency, for example) and several subharmonics of an infinitesimal magnitude, Eq. (45) leads to a set of coupled, linear, first-order differential equations. These equations show that there is a transfer of energy from the blade passage to the subharmonics. Once these subharmonics grow to any appreciable magnitude, the equations become nonlinear. Even then the amplitudes obey the conservation principle

$$\sum_{j=1}^{\infty} j^2 A_j{}^2=\text{constant} \tag{50}$$

In the next section, the effect of various design parameters on the generation of subharmonics is discussed. In particular it will be seen that an absorbent liner with admittance ratio of the order of ϵ has a significant effect on the process of generation.

D. Numerical Simulation

We simulated a fan with 8 blades radiating in a narrow annulus. It was assumed that the tip Mach number was 1.2. Initially the level of absorption was assumed to be zero. Near the inlet, the blade-passage radiation was assumed to occur in the $n=8$ mode at 8 times Ω, the shaft rotational frequency. The radiation at Ω was assumed to occur in the form of $n=1$ at a level considerably below that of the blade-passage frequency. The radiation at 2Ω was again assumed to occur in a two-lobe pattern at the same level as that of $n=2$, and so on. The initial level at the blade-passage frequency was 150 dB and that of the subharmonics was 115 dB. With these inputs, Eq. (45) was solved by means of a modified Runge-Kutta procedure. The results at $z=75h$ are seen in Fig. 3a. The blade-passage frequency has gone down and the subharmonics have grown in size. Figure 3b shows the results from the modified equation (51), incorporating the effect of absorbents. In this case, a substantial suppression of the MPT's is observed.

As a "qualitative" comparison, consider the data given by Benzakein. Details of blade spacing, treatment, etc. are not matched. What is interesting is that a reduction in the MPT level is indeed observed due to the lining of the inlet.

Our model thus demonstrates a mechanism for both the generation and the suppression of the MPT's. Considerable refinement would be needed, on the lines of the observations in the next section, if a quantitative prediction model is to be developed.

V. The Influence of Various Design Parameters

A. Effect of Geometry

Figure 2 shows the dependence of s_j on ϵ_2. For a very narrow annulus, $\epsilon_2 \to 0$ and $s_j \to 1$. For a larger annulus, ϵ_2 s_j decreases. s_j depends on both the circumferential and radial mode numbers. The hub-to-tip ratio is defined as $R=1/(1+\epsilon_2)$. In general, the larger the annulus the weaker the resonance.

B. Influence of Radial Modes

There is another effect to be seen when the annulus is not very narrow. The dispersion relationship changes to $m^2\pi^2+\epsilon_2^2 n^2+\alpha^2=\omega^2$. This relationship implies that α and, therefore, the cutoff frequency depend on the radial modes m. It can be seen that those modes with identical ratio of m and n resonate. Thus (4,8) mode will resonate with (2,4) and (1,2) modes.

Higher m implies a more uneven radial distribution. Except near the tip speed of unity, one could assume that higher radial modes are produced in decreasingly smaller strengths. Thus if n for the blade-passage mode, which is the same as B, is kept a prime number, the only possible subharmonics would be those that get coupled with the $m=0$ mode. Otherwise, the radial modes create additional cutoff frequencies near which β gets large. All these frequencies correspond to $M_T>1$. For sufficiently high tip Mach number, however, the field of a rotor is more nearly uniform and only $m=0$ is being generated. β reduces farther away from $M_T=1$ and therefore the nonlinear interaction and MPT generation has a tendency to drop at higher tip numbers.

C. Influence of Absorbents

If the duct is soft, the interaction (44) should be modified to

$$\frac{\mathrm{d}A_j}{\mathrm{d}z}+f_jA_j+\lambda_j(E_j-G_j)=0 \quad (j=1,2,\ldots) \tag{51}$$

where the dissipation coefficients f_j are defined as

$$f_j=\frac{\omega R_0}{\alpha(R_0^2+j^2\chi_0^2)} \tag{52}$$

Let us define an admittance ratio

$$K_t(j) = \frac{R_0}{R_0^2 + j^2 \chi_0^2} \tag{53}$$

When $j = 1$, this will coincide with the measured admittance ratio of the material. For small values of admittance, the greater the admittance is, the greater the absorption. For higher harmonics, $K_t(j)$ decreases. For this reason the lower subharmonics are marginally easier to suppress. This might suggest that a more uniformly effective liner has χ_0 smaller than R_0 so that the fluctuations in K_t with j, the harmonic number, are reduced.

If K_t is much larger than the order of ϵ, it can be shown that the governing equations for the amplitudes become uncoupled and linear. A strong absorbent renders the problem linear and therefore suppresses the generation of subharmonics. An absorbent with admittance of the order of ϵ would render the nonlinear interactions weaker. Effectiveness of a liner with an absorbent is amply seen in Fig. 3. This is in agreement with experimental data such as the one shown in Fig. 4.

D. Influence of Shear

The influence of shear is to complicate the form of the Eq. (45). Details are given in Ref. 5. A linear shear profile of small slope has no influence. A fully developed steady shear profile reduces nonlinearity and once the order of vorticity is larger than ϵ, the resonance is fully suppressed.

E. Effect of Stators

For a rotor with B blades and a stator with S vanes, Tyler and Sofrin have shown that the radiation is at $n\Omega B$ rad/sec, but the circumferential lobe number is

$$\bar{n} = nB \pm lS \tag{54}$$

where l could be 0, 1, 2, 3, ... The cutoff can occur below or above the supersonic tip speed condition. Additional resonances are possible. Some of these would coincide with the frequencies of single rotor, especially if S, the number of stators, and B have a large enough common factor.

The negative sign in Eq. (51) suggests that a stator can lower the critical frequency and cause possible MPT's at $M_T < 1$ if S and B indeed have a large common factor.

F. Effect of the Axial Length

It turns out that for large values of $\lambda_j \bar{z}/\bar{a}$, the problem becomes highly stochastic. Even a slight change in the initial conditions leads to large changes in the output. (This was also observed in an experimental simulation.[5]) Under these conditions it really doesn't matter whether the blades are nonuniformly spaced or not. We would see the classic case of the self-induced oscillatory phenomenon.

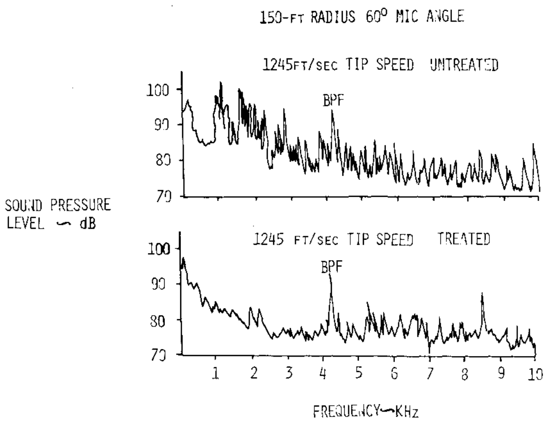

Fig. 4 Effect of absorbents on the MPT's.

This conclusion is in line with the one drawn by Goel et al.[13] in their study of the generalized Volterra equations. Equation (45) is a particular example of these equations. What is perhaps more interesting is that Kraichnan,[14] in his study of turbulence, has obtained an equation similar to Eq. (45). His study would represent vorticity-vorticity type of nonlinear interactions. Ours represents the acoustic-acoustic interactions. In both these cases, for large enough values of the relevant parameters, the problems become stochastic.

VI. Practical Means of Suppression of MPT's

Practical means of control of MPT's would depend on two strategies: 1) reduce λ_j, and 2) increase absorbents where λ_j is large. The technique could be outlined as follows:

Step 1

Identify the design conditions under which λ_j is likely to be large. The most important region is near the tip Mach number of unity, in the absence of stators or inflow distortions. λ_j is also going to be large upstream and in a narrower part of the annulus. Vorticity reduces the downstream λ_j and the shape factor s_j goes down as the annulus becomes bigger.

Step 2

Avoid designs that would have M_T in the critical range. Also avoid having the number of rotor blades and stator blades share a large common factor. In particular, even for a single rotor, a prime number of blades is to be preferred.

Step 3

Those parts of the inlet that lead to high λ_j should be treated with a sound-absorbent material. Perhaps, more uniform suppression of the subharmonics would be obtained by using resistive liners. More admittance is needed at higher intensity. Make sure to cover the entire region of potentially high λ_j. In an inlet, this would be the part containing the centerbody and forming an annulus.

VII. Conclusions

A mechanism of generation and suppression of the multiple pure tones in aeroengine ducts has been suggested. It is shown that a "strong interaction" takes place between the subharmonics, which initially have small amplitudes, and the blade-passage tone, possessing a large amplitude. This interaction leads to a transfer of energy to the lower harmonics, as opposed to the upward cascading of energy due to the conventional "weak interactions." Strong interactions occur when the system is in resonance. It has been found that the spinning modes in a narrow annulus are indeed in a state of a nonlinear resonance. The governing equations describing these interactions, and the dependence of the relevent parameters on various design parameters, have been described. Numerical simulation, using the governing equations, shows an excellent qualitative agreement with the observed phenomenon.

This analysis has led to several new suggestions for the efficient suppression of the MPT's. The technique therefore possesses a considerable advantage over the conventional technique of analyzing the MPT phenomenon by means of the weak-shock theory.

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